Anomalies without Massless Particles

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Abstract

Baryon and lepton number in the standard model are violated by anomalies, even though the fermions are massive. This problem is studied in the context of a two dimensional model. In a uniform background field, fermion production arise from non-adiabatic behavior that compensates for the absence of massless modes. On the other hand, for localized instanton-like configurations, there is an adiabatic limit. In this case, the anomaly is produced by bound states which travel across the mass gap. The sphaleron corresponds to a bound state at the halfway point.

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1. Introduction

The divergence of lepton and baryon currents in the Standard Model is independent of the fermion masses. For a single family, the baryon and lepton number anomaly is

$$\partial_\mu J^\mu_B = \partial_\mu J^\mu_L = \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}(W^{\mu\nu}W^{\alpha\beta}) - \frac{g'^2}{64\pi^2}\epsilon_{\mu\nu\alpha\beta}B^{\mu\nu}B^{\alpha\beta}\ , \tag{1.1}$$

where $W^{\mu\nu}$ is the $SU(2)$ field strength and $B^{\mu\nu}$ is the $U(1)$ field strength. This differs greatly from the axial current equations of Q.E.D. because in Q.E.D. the production of axial charge depends critically on whether or not the electron is massive. I will begin by reviewing the reasons for this sensitivity. Then I will show why these reasons are not applicable to a spontaneously broken theory with a vector current anomaly, such as the standard model. The results give some insight into the production of baryon number in the standard model by sphalerons, which has been of much recent interest.

The divergence of the axial current in Q.E.D. [1] is

$$\partial_\mu \overline{\psi}\gamma^\mu\gamma^5\psi = \frac{e^2}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + 2im\overline{\psi}\gamma^5\psi. \tag{1.2}$$

In a background gauge field the matrix element of the last term is

$${}_A\left\langle 0\,out\right|2im\overline{\psi}\gamma^5\psi\left|0\,in\right\rangle = -\frac{e^2}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + \ldots \tag{1.3}$$

The remaining terms are higher dimension functions of the gauge fields and vanish in an adiabatic aproximation. If the electron is massive then there is no axial charge violation in an adiabatic approximation because the first and last terms in equation (1.2) cancel. This cancellation is obvious from the start if one calculates the anomaly using a Pauli Villars regulator field. Then the regulated axial current satisfies

$$\partial_\mu J^{5\mu}_r = 2i\Lambda\overline{\chi}\gamma^5\chi + 2im\overline{\psi}\gamma^5\psi\ , \tag{1.4}$$

where χ is the regulator field and Λ is its mass. χ is bosonic, so χ loops have the opposite sign from ψ loops. Therefore there can be no mass independent terms in the matrix element of $\partial_\mu J^{5\mu}_r$ in a background gauge field.

This cancellation also has a simple spectral interpretation. An explanation of the Q.E.D. axial anomaly based upon the spectrum of a massless electron in a background magnetic field has been given by Nielson and Ninomiya [2]. Their arguments are briefly summarized below. Consider a uniform background magnetic field in the z direction. In

the massless case, positive and negative chirality fermions decouple, so there are two sets of Landau levels. The positive and negative chirality Landau levels contain zero-modes with $E = -p_z$ and $E = +p_z$ respectively. Suppose one turns on a positive uniform electric field $\mathcal{E}$ in the z direction. In an adiabatic approximation, solutions flow along spectral lines according to the Lorentz force law $\frac{dp}{dt} = e\mathcal{E}$. Thus right chiral zero-modes slide out of the Dirac sea while left chiral zero-modes slide deeper into the Dirac sea (fig. 1). This motion produces a net axial charge but no electric charge. By a careful counting of states one reproduces the global form of the anomaly

$$\frac{dQ^5}{dt} = \mathrm{V}\frac{e^2}{2\pi^2}\mathcal{E}_z\mathcal{B}_z \ , \tag{1.5}$$

where V is the volume of space. Now consider the same background fields but suppose the electron is massive. In this case, there are no zero-modes among the Landau levels. In the absence of zero-modes adiabatic evolution just maps the Dirac sea into itself, so axial charge can not be adiabatically generated.

The discussion above is not applicable to the standard model because standard model fermions can be given masses without changing the baryon or lepton number violation in fixed gauge field background. Dirac mass terms do not carry vector charge, so they do not effect the divergence of a vector current. Yet in an adiabatic limit it seems that presence or absence of mass terms *must* effect the divergence of a current. In the following, this paradox will be resolved by solving the equations of motion for certain background fields which, according to the anomaly equation, should generate charge. I will demonstrate that spatially uniform backgrounds which generate vector charge have no adiabatic limit. Such backgrounds produce the anomaly by causing hopping between energy levels. On the other hand, localized instanton-like backgrounds do possess an adiabatic limit. Backgrounds of this type will be shown to produce the anomaly via fermionic bound states whose energies traverse the gap between $E = -m$ to $E = m$. This give a better understanding of the mechanism of baryon number production in the standard model by sphalerons. The sphaleron configuration corresponds to the half-way point with a zero energy bound state.

Because of the chiral couplings, the standard model Landau levels are quite complicated. To avoid calculating Landau levels in $3+1$ dimensions, I will instead consider a spontaneously broken $U(1)$ axial gauge theory in $1+1$ dimensions. While the details of the computation are different, many of the results obtained in $1+1$ dimensions are expected to hold in $3+1$ dimensions. The lagrangian of this theory is

$$\mathcal{L} = -\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ + A\!\!\!/\gamma^5 - \lambda\phi^* P_L - \lambda\phi P_R)\psi + \frac{1}{2}D^\mu\phi^* D_\mu\phi - U(\phi^*\phi) \ . \tag{1.6}$$

This simplified model possesses the two traits whose consistency I wish to demonstrate; a massive spectrum and a mass independent vector current divergence,

$$\partial_\mu \overline{\psi} \gamma^\mu \psi = \frac{g}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu} \ . \tag{1.7}$$

For the moment I will not consider the full dynamical theory, but only that given by

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ + gA\!\!\!/(x)\gamma^5 - \lambda\rho(x)e^{i\theta(x)\gamma^5})\psi \ , \tag{1.8}$$

where $\rho(x) = v$ asymptotically. It should be possible to demonstrate the anomaly by considering the momentum space equations of motion, as was done for massless Q.E.D. by Nielsen and Ninomiya using the Lorentz force law. A few remarks are in order about how to do this. Let the Dirac field in a background be expanded as follows:

$$\psi(x,t) = \int \frac{dp}{2\pi} c_{p,i}(t) u_{p,i} e^{ipx} \ , \tag{1.9}$$

where $u_{p,i}$ are free massive spinors normalized to 1, and the index i distinguishes between positive and negative frequency solutions when the backgrounds vanish. All the background dependance is contained in the time evolution of $c_{p,i}(t)$ When the backgrounds vanish,

$$c_{p,i}(t) = \exp(i\omega_{p,i}t)c_{p,i}(0) \ , \tag{1.10}$$

where

$$\omega_{p,\pm} = \pm\sqrt{p^2+m^2} \ . \tag{1.11}$$

Given a knowledge of which states are occupied at an initial time, one can determine which states are occupied at a final time by looking at the evolution of the coeficients $c_{p,i}$. At this point however, the use of this expansion to determine the vector charge or the particle number is very ambiguous. One can make transformations of ψ, corresponding to certain transformations of the background fields, which change the $c_{p,i}$. For example transformations exist which map something that looks like the Dirac sea into something that looks like an excited state with non zero vector charge. An invariant definition of charge is needed. Such a definition must depend on the background fields as well as the Fourier coefficients. In order to make the computation of the charge simple, I will only consider processes in which local gauge invariant functions of the background fields vanish at asymptotic times. This means that the initial and final θ and A^μ are gauge equivalent to $\theta = 0$ and $A^\mu = 0$. In this case the proper definition of charge at asymptotic times

is simple. In Dirac sea language, one subtracts the number of vacant negative frequency states from the number of occupied positive frequency states. The occupation number of a positive or negative frequency state of momentum p is proportional to $|c_{p,\pm}|^2$ in the gauge in which the backgrounds vanish. Equivalently, in second quantized language one can adopt a normal ordered definition of charge at asymptotic times. The change in the charge can then be written in terms of Bogolubov coefficients relating the operators $\hat{c}_{p,i}$ in the asymptotic past to those in the asymptotic future, where these operators are defined in the gauge in which the backgrounds vanish. Note that at intermediate times the gauge invariant backgrounds do not vanish so a well defined Bogolubov transformation between asymptotic past and intermediate times does not exist. Normal ordering is no longer sensible at intermediate times because solutions can not be classified as positive or negative frequency. However, I will never explicitly calculate the charge at intermediate times *.

2. Uniform Backgrounds

In the spirit of the anomaly calculations done by Nielsen and Ninomiya, I will first consider a process in which a spatially uniform axial electric field is turned on and then off. I will also choose a uniform (spatially parallel transported) Higgs field background. The particular background to be considered is

$$A^0 = -\mathcal{E}(t)x, \quad A^1 = 0, \quad \theta = 0 \ , \tag{2.1}$$

where $\mathcal{E}(t) = \mathcal{E}$ for $0 < t < T$ and 0 at all other times. In this gauge, with the initial and final backgrounds vanishing, the coefficients $c_{p,i}$ have an immediate interpretation in terms of particle and charge production. Due to the axial electric field, vector charge generation is expected, and should be evident in the time evolution of these coeficients. The equations of motion for $c_{p,i}(t)$ are complicated at low p, but simplify greatly at large $|p|$. The simplification occurs because, as one would expect, the fermion mass can be neglected

* At intermediate times the charge is defined by axial gauge invariance and charge conjugation symetry. For example one can use an axially gauge invariant point split charge which is odd under charge conjugation. When the gauge fields vanish this is equivalent to the usual normal ordered definition of charge.

at large $|p|$. A straightforward calculation gives the equation describing behavior deep in the Dirac sea:

$$\begin{aligned}\frac{d}{dt}c_{p,-}(t) =& \frac{gE}{\pi}u^{\dagger}_{p,-}\gamma^5 u_{p,-}\frac{d}{dp}c_{p,-}(t) - i\omega_{p,-}c_{p,-}(t) \ , \\ =& \frac{p}{|p|}\frac{gE}{\pi}\frac{d}{dp}c_{p,-}(t) - i\omega_{p,-}c_{p,-}(t) \ .\end{aligned} \tag{2.2}$$

This equation is not complete, but the neglected terms are all supressed by factors of $\frac{m}{|p|}$. The solution is

$$c_{p,-}(t) = c_{\tilde{p},-}(0)\exp(-i\omega_{p,-}t) \ , \tag{2.3}$$

where

$$\tilde{p} = p - \frac{p}{|p|}\frac{gE}{\pi}t \ , \tag{2.4}$$

which is easily recognized as an axial version of the Lorentz force law. Therefore states along the negative frequency spectral lines at large $|p|$ flow inward towards small $|p|$. Because of unitarity and Fermi statistics, solutions can not pile up at small $|p|$. Therefore there must be level hopping at small $|p|$. Positive frequency states must appear at a rate matching the inward flow of negative frequency states across some large $|p|$ cutoff (fig. 2). I thus arrive at the result that the backgrounds of (2.1) have no adiabatic limit. Therefore the absence of zeromodes has no effect on charge production. Putting the system on a line of length L with periodic boundary boundary conditions on the Fermi field, one finds that the number of states crossing the cutoff per unit time is $\frac{L}{\pi}gE$. This yields the expected anomaly $\frac{1}{L}\frac{dQ}{dt} = \frac{gE}{\pi}$.

There is actually no reason to expect adiabatic behavior with uniform backgrounds. The backgrounds of (2.1) have singular time dependence when the electric field is turned on or off. One can make the time dependence of these backgrounds nonsingular either by smoothly switching the electric field on and off, or by going to $A^0 = 0$ gauge. If one does the former, one can try to make the backgrounds vary slowly in time by having the electric field $\mathcal{E}(t)$ vary slowly in time. However, no matter how slowly the electric field varies, A^0 will vary rapidly at large distances since $A^0 = -\mathcal{E}(t)x$. In $A^0 = 0$ gauge, the backgrounds of (2.1) become

$$\begin{aligned} A^1 =& 0, \quad \theta = 0 & & t < 0 \ , \\ A^1 =& \mathcal{E}t, \quad \theta = -2\mathcal{E}xt & \text{for} \qquad & 0 < t < T \ , \\ A^1 =& \mathcal{E}T, \quad \theta = -2\mathcal{E}xT & & T < t \ . \end{aligned} \tag{2.5}$$

One can try to make these backgrounds vary slowly in time by making $\mathcal{E}$ small. Yet, no matter how small $\mathcal{E}$ is, the Higgs phase θ winds wildly with time at large distances.

Therefore the non adiabatic nature of uniform charge producing backgrounds is an infinite volume effect.

It is actually easy to see the low momentum level hopping explicitly without invoking Fermi statistics. In $1+1$ dimensions $\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu$. One can use this fortuitous fact to solve the equations of motion at all momenta. For $0 < t < T$ the background fields of (2.1) are equivalent to a a background vector gauge field * with $V^0 = 0$ and $V^1 = -\mathcal{E}x$. The vector field strength vansishes, so the time evolution of ψ at intermediate times is trivial. $\psi' \equiv e^{\frac{i}{2}\mathcal{E}x^2}\psi$ evolves as a free field:

$$c'_{p,i}(t) = e^{-i\omega_{p,i}t}c'_{p,i}(0) \ , \tag{2.6}$$

One only has to transform back from c' to c to get $c_{p,i}(t)$ as a function of the initial coefficients $c_{r,l}(0)$. The result is that

$$c_{p,i}(t) = \sum_l \int \frac{dr}{2\pi} T_{p,i;r,l} c_{r,l}(0) \ , \tag{2.7}$$

where

$$T_{p,i;r,l} = \frac{2\pi}{E} u^\dagger_{p,i} \left[\sum_j \int dq \exp(i\frac{p-r}{E}q - i\omega_{q,j}t) u_{q,j} u^\dagger_{q,j} \right] u_{r,l} \exp(-i\frac{p^2-r^2}{2E}) \ . \tag{2.8}$$

The quantity within the brackets can be written

$$\frac{i}{2}\gamma^0(i\partial\!\!\!/ - m) \langle 0| \left[\phi(z), \phi(0)\right] |0\rangle \ , \tag{2.9}$$

where ϕ is a massive free scalar field in $1+1$ dimensions, and $(z^0, z^1) \equiv (t, \frac{p-r}{E})$. The "light cone" singularity in (2.9) gives the leading term of (2.8) :

$$T_{p,i;r,l} = \frac{\pi}{E}\overline{u}_{p,i} \left(i\gamma^+ \delta(z^+) + i\gamma^- \delta(z^-) \right) u_{r,l} \exp(-i\frac{p^2-r^2}{2E}) + \ldots \tag{2.10}$$

Let us rewrite this in a form which is easier to interpret:

$$\begin{aligned} T_{p,i;r,l} = \frac{2\pi}{E} & \left[u^\dagger_{p,i} \frac{1-\gamma^5}{2} u_{r,l} \delta(t + \frac{p-r}{E}) + u^\dagger_{p,i} \frac{1+\gamma^5}{2} u_{r,l} \delta(t - \frac{p-r}{E}) \right] \\ & \times \exp(-i\frac{p^2-r^2}{2E}) \end{aligned} \ , \tag{2.11}$$

* This method of solving the Dirac equation brings up a troubling question. If an axial gauge field background can generate vector charge, then apparently a vector gauge field background can also generate vector charge. I discuss why this last statement is not true in appendix A.

where I have used the fact that $\gamma^0\gamma^{\pm} = 1 \pm \gamma^5$ in $1+1$ dimensions. The axial Lorentz force law is clearly visible in the delta functions and the associated left or right chiral projectors. The low momentum level hopping is also manifest. The hopping of negative frequency to positive frequency states is described by $T_{p,+:r,-}$. At large p and r of the same sign, the spinors $u_{p,+}$ and $u_{r,-}$ have opposite chirality so that $u^\dagger_{p,+}\frac{1\pm\gamma^5}{2}u_{r,-}$ vanishes. Thus in the limit of large momenta at fixed time, $T_{p,+;r,-}$ vanishes. However at small $|p|$ the spinors have mixed chirality so that (2.11) does not vanish when $p - r = \pm\mathcal{E}t$, and the predicted level hopping occurs. It is interesting to note that factor (2.11) is almost the transformation function associated with an axial transformation of ψ:

$$\psi(x) \to \psi'(x) = \exp(-iEtx\gamma^5)\psi(x) \ , \tag{2.12}$$

is equivalent to

$$c_{p,i} \to c'_{p,i} = \sum_l \int \frac{dr}{2\pi} T'_{p,i;r,l} c_{r,l} \ , \tag{2.13}$$

where

$$T'_{p,i;r,l} = T_{p,i;r,l} \exp(-i\frac{p^2 - r^2}{2E}) \ . \tag{2.14}$$

This is not to be confused with an axial *gauge* transformation because the initial and final background fields are the same; $A^\mu = 0$ and $\theta = 0$. An axial gauge transformation does nothing, but an axial transformation which leaves the Higgs and gauge potentials unchanged can produce particles and vector charge. This should be no surprise given the bosonization rules [3] for an axial gauge theory in $1+1$ dimensions. The vector charge density in bosonized form is

$$J^0 = \frac{1}{\sqrt{\pi}}(\partial_1\chi - \frac{1}{\sqrt{\pi}}A^1) \ , \tag{2.15}$$

where χ is the bosonic counterpart to ψ. An axial transformation

$$\psi \to e^{if(x)\gamma^5}\psi \ , \tag{2.16}$$

corresponds to

$$\chi \to \chi + \frac{1}{\sqrt{\pi}}f(x) \ . \tag{2.17}$$

Therefore an axial transformation of the type (2.12) above produces a net vector charge.

3. Localized Backgrounds

The uniform backgrounds of (2.1) are interesting but perverse because the gauge invariant objects built from the Higgs and gauge fields do not fall off at large spatial distances. Furthermore these configurations can exist only in an infinite volume because they are inconsistent with periodic boundary conditions. Therefore let us instead consider localized, charge producing backgrounds. By localized, I mean that the energy density carried by the backgrounds is at its minimum outside a spacetime disc of finite radius. At fixed ΔQ one can always make such backgrounds vary arbitrarily slowly in time, so that there is no argument against the existence of an adiabatic limit. We are again confronted with the puzzle of how vector charge can be produced by a weak electric field in a theory with a gap.

The clue to the puzzle is that one can not go to unitary ($\theta = 0$) gauge from localized backgrounds which produce charge. For such backgrounds $D_\mu \phi = 0$ asymptotically. Therefore

$$\oint dx^\mu \partial_\mu \theta = -2g \oint dx^\mu A_\mu = -2\pi \Delta Q \ . \tag{3.1}$$

If ΔQ is not zero, then $\phi^* \phi$ must vanish somewhere due to the non vanishing Higgs winding number. In the presence of such a defect there may be a bound state as well as the continuum of "scattering" solutions with $E = \pm\sqrt{p^2 + m^2}$. In an adiabatic limit the only way charge can appear is if a bound state traverses the mass gap. As the defect is created and destroyed in a process with $\Delta Q = 1$, the bound state energy should change continuously from $-m$ to m. I will show that this is indeed the case. The sphaleron corresponds to a bound state at the half-way point and has charge one half [4].

An example of a localized configuration giving $\Delta Q = 1$ is

$$\begin{aligned} \phi =& v \exp\left(i\alpha(t) \frac{x}{|x|} \right) \ , \\ A^0 =& -\frac{1}{2g} \frac{x}{|x|} \frac{d\alpha}{dt} \ , \\ A^1 =& 0 \ , \end{aligned} \tag{3.2}$$

where the phase $\alpha(t)$ rotates by a total angle of $-\pi$ from $\alpha(-\infty) = 0$ to $\alpha(\infty) = -\pi$. In an adiabatic limit $\alpha(t)$ varies slowly and the gauge fields can be neglected. The defect at $x = 0$ is spatially pointlike for convenience; For a fixed α, finding the spectrum is a trivial matching problem. (A less singular version of this background is drawn in fig. 3) One finds

a set of scattering solutions with $E = \pm\sqrt{p^2 + m^2}$, but there is also a bound state solution with $E^2 < m^2$. Continuity of the solution across $x = 0$ requires

$$e^{-2i\alpha} = \frac{E + i\sqrt{m^2 - E^2}}{E - i\sqrt{m^2 - E^2}} \ . \tag{3.3}$$

This yields a bound state with energy $E = -m\cos\alpha$. As α varies adiabatically from 0 to $-\pi$, a single bound charge is carried across the gap. Note that this alone does not guarantee the net production of charge. A bound state could travel across the gap and leave a negative energy hole. The axial Lorentz force law causes negative frequency states to slide inwards towards zero momentum, which prevents the appearance of a hole. In an adiabatic approximation, the gauge fields are negligible pertubations on the spectrum, but drive the spectral flows needed to produce the anomaly.

For more general localized backgrounds, an index theorem enables one to count the number of time dependent energy eigenvalues which travel across the gap. Consider spinor functions $f(x,\tau)$ anihilated by the operator

$$\hat{D} \equiv \frac{\partial}{\partial\tau} + \hat{H}(\tau) \ , \tag{3.4}$$

where by varying the parameter τ from $-\infty$ to ∞ one goes slowly through the same cycle of Dirac hamiltonians $\hat{H}$ that occur in real time. I will write the energy eigenvalues as $E_n(\tau)$ and the energy eigenfunctions as $\chi_n(x,\tau)$. Since $\hat{H}(\tau)$ is a slowly varying function of τ, the solutions of equation (3.4) can be written as

$$f(x,\tau) = a_n(\tau)\chi_n(x,\tau) \ , \tag{3.5}$$

where there is no sum on n and

$$a_n(\tau) = a_n(0)e^{-\int_0^\tau d\tau' E_n(\tau')} \ . \tag{3.6}$$

This solution is only normalizable if $E_n(\tau)$ has a negative value at $\tau = -\infty$ and a positive value at $\tau = +\infty$. Now consider the adjoint operator

$$\hat{D}^\dagger \equiv -\frac{\partial}{\partial\tau} + H(\tau) \ . \tag{3.7}$$

A function $a_n(\tau)\chi_n(x,\tau)$ annihilated by $\hat{D}^\dagger$ is only normalizable if $E_n(\tau)$ has a positive value at $\tau = -\infty$ and a negative value at $\tau = +\infty$. Hence the total charge generated by bound states crossing the gap is equal to the difference in the number of normalizable

modes annihilated by $\hat{D}$ and the number of normalizable modes annihilated by $\hat{D}^{\dagger *}$. This quantity is known as the index of $\hat{D}$. The operator whose index I wish to calculate is

$$\hat{D} = \frac{\partial}{\partial\tau} + \gamma^0 \left(i\gamma^1(\partial_1 + igA_1\gamma^5) - \lambda\phi(x,\tau)\frac{1+\gamma^5}{2} - \lambda\phi^*(x,\tau)\frac{1-\gamma^5}{2} \right) , \tag{3.8}$$

where asymptotically

$$\phi = ve^{i\theta} . \tag{3.9}$$

A^0 is absent from $\hat{D}$ because it is negligible in an adiabatic approximation. One can take the adiabatic limit of a process with fixed ΔQ by making the following gauge invariant rescaling of the fields:

$$\begin{aligned} \phi'(x,t) =& \phi(x, \frac{t}{\lambda}) , \\ A'^0(x,t) =& \frac{1}{\lambda} A^0(x, \frac{t}{\lambda}) , \\ A'^1(x,t) =& A^1(x, \frac{t}{\lambda}) . \end{aligned} \tag{3.10}$$

In the large λ limit A^0 vanishes. A^1 is a nonvanishing adiabatic parameter, but one can gauge it to zero. Doing so effects only the eigenfunctions of $\hat{H}(\tau)$ but not the eigenvalues. A straight-forward method to calulate the index of Dirac operators on R_n has been constructed by Weinberg[6]. Using these methods, the index of $\hat{D}$ with $A^1 = 0$ is found to be *

$$\frac{1}{2\pi} \oint dx^\mu \partial_\mu \theta , \tag{3.11}$$

which is gauge invariant. This is just as one expects given equation (3.1).

The relation of this index theorem to charge production can also be understood in terms of the euclidean path integral using methods due to Fujikawa [8] and 't Hooft [9]. The fermionic portion of the partition function is

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \exp\left(-\int d^2x \overline{\psi}\hat{K}\psi \right) , \tag{3.12}$$

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* Witten has applied similiar methods to a different problem [5].

* Weinberg applied his methods to count the number of zero energy modes of a vortex-fermion system in 2 spatial dimensions. This system was previously considered by Jackiw and Rossi [7] who suggested the existence of an index theorem equating the number of fermion zero energy modes to the vortex number. The index theorem for their model is very similiar to the one considered in this paper.

where

$$K = \gamma^0(\partial_0 - \hat{H}). \tag{3.13}$$

Let ψ and $\overline{\psi}$ be expanded as

$$\begin{aligned}\psi(x) =& \sum_n a_n f_n(x) \ , \\ \overline{\psi}(x) =& \sum_l \overline{b}_l g_l^\dagger(x) \ ,\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\hat{K}^\dagger \hat{K} f_n(x) =& \lambda_n f_n(x) \ , \\ \hat{K}\hat{K}^\dagger g_l(x) =& \alpha_l g_l(x) \ ,\end{aligned} \tag{3.15}$$

and $f_n(x)$ and $g_l(x)$ are normalized to one. There is a one to one mapping between eigenfunctions of $\hat{K}^\dagger\hat{K}$ and $\hat{K}\hat{K}^\dagger$ provided that the eigenvalue is not zero. $\hat{K}$ maps eigenfunctions of $\hat{K}^\dagger\hat{K}$ into eigenfunctions of $\hat{K}\hat{K}^\dagger$ with the same non zero eigenvalue, while $\hat{K}^\dagger$ does the inverse mapping. However if $\hat{K}^\dagger\hat{K}f(x) = 0$ or $\hat{K}\hat{K}^\dagger g(x) = 0$, then there is no mapping because $\hat{K}\hat{K}^\dagger f(x) = 0$ implies that $\hat{K}f(x) = 0$, and $\hat{K}^\dagger\hat{K}g(x) = 0$ implies that $\hat{K}^\dagger g(x) = 0$. The difference between the number of zeromodes of $\hat{K}^\dagger\hat{K}$ and $\hat{K}\hat{K}^\dagger$ is given by the index of $\hat{K}$. A zeromode of either $\hat{K}^\dagger\hat{K}$ or $\hat{K}\hat{K}^\dagger$ contributes nothing to the euclidean action. Therefore the integral over the grassman coefficient of a zeromode will vanish unless the coefficient appears in the expansion of an operator in a Green's function. It is easy to see from this that the contributions of a given Higgs and gauge field background to a Green's function vanishes except when the number of ψ's in the Green's function differs from the number $\overline{\psi}$'s by the index of $\hat{K}$. For example, if $\hat{K}^\dagger\hat{K}$ has one zeromode $f_0(x)$ and $\hat{K}\hat{K}^\dagger$ has no zeromode, then

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\psi(x) \exp\left(-\int d^2y\overline{\psi}\hat{K}\psi\right) = \sqrt{det\hat{K}\hat{K}^\dagger} f_0(x) \ . \tag{3.16}$$

In general the net vector charge produced is given by the index of $\hat{K}$, which in an adiabatic limit is the same as the index of $\hat{D}$ because the two operators differ only by a factor of γ^0. The connection between the spectral and path integral approaches to the anomaly is now clear *.

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* This connection is not novel. The relation between modes annihilated by the Euclidean Dirac operator and spectral flows which take states in and out of the Dirac sea was discussed by Nielsen and Ninomiya in the context of massless fermions [2].

An interesting feature of the index theorem for a spontaneously broken axial theory is that it permits Higgs and gauge field backgrounds to create single fermions and not just pairs. The Euclidean equations of motion possess a symetry $\psi \to \gamma^0\psi^*$. In the absence of the Higgs coupling to fermions, $\hat{K}$ anticommutes with γ^5, so zeromodes can be chosen to be chiral. Therefore in the massless axial theory zeromodes occur in pairs of opposite chirality which are related by the above symetry. This pairing is a reflection of Q_5 conservation. However in the spontaneously broken axial theory, Q_5 has a Higgs component as well as a fermionic component, and only the sum is conserved. It is no longer true that $\{\hat{K}, \gamma^5\} = 0$. Therefore zeromodes can no longer be chosen to be chiral. In fact, in an adiabatic approximation one can prove that the mapping $\psi \to \gamma^0\psi^*$ does not yield independent solutions. This is done in appendix B. The production of single fermions by a background is not a violation of gauge or Lorentz invariance. For example a single fermion can not get a vacuum expectation value because the path integral over gauge and Higgs fields in the one instanton sector vanishes, even if the fermionic integral does not.

4. Dynamics

So far it has only been demonstrated how charge violation proceeds independently of the fermion masses in the case of background Higgs and gauge fields. I will now show how this works in the dynamical case. This will be done by demonstrating the consistency of the Ward identities with a massive spectrum. Similiar results should hold for three current correlation functions in $3+1$ dimensions.

The current equations are

$$\partial_\mu J_5^\mu = \partial^\mu(\overline{\psi}\gamma_\mu\gamma_5\psi + i\phi^* \overleftrightarrow{D_\mu} \phi) = 0 \ , \tag{4.1}$$

and

$$\partial_\mu J^\mu = \partial_\mu \overline{\psi}\gamma^\mu\psi = \frac{1}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ . \tag{4.2}$$

A simple path integral manipulation relates the current equations to Ward idendities for $\langle 0|\, T^* \left(J^\mu(x)\ J_5^\alpha(y)\right) |0\rangle$. One finds that

$$\frac{\partial}{\partial y^\alpha} \langle 0|\, T^* \left(J^\mu(x)\ J_5^\alpha(y)\right) |0\rangle = 0 \ , \tag{4.3}$$

and

$$\frac{\partial}{\partial x^\mu} \langle 0| T^* \left(J^\mu(x)\ J_5^\alpha(y)\right) |0\rangle = \frac{1}{\pi}\epsilon^{\mu\alpha}\frac{\partial}{\partial x^\mu}\delta(x-y) + \frac{1}{\pi}\langle 0| T^* \left(\epsilon_{\mu\nu}\partial^\mu A^\nu(x)\ J_5^\alpha(y)\right) |0\rangle \ . \tag{4.4}$$

If it were not for the last term in (4.4), the two Ward identities (4.3) and (4.4) would ensure the existence of a massless pole in the current correlator [10]. Naively one might expect the last term in (4.4) to give at most $\mathcal{O}(g)$ perturbative corrections to this pole or its residue.

We are thus confronted with the same dilemma as before. The massive spectrum of a spontaneously broken U(1) axial gauge theory appears to be inconsistent with its vector current anomaly. The resolution of the puzzle lies in the fact that the gauge boson mass is proportional to g. It turns out that the last term in (4.4) contains an order zero piece which exactly cancels the first term at small p^2. The last term in (4.4) can be rewritten as

$$-\frac{1}{2\pi}\langle 0| T^* \left(\epsilon_{\mu\nu}\partial^\mu A^\nu(x)\ 2v^2\left(\partial^\alpha\theta(y) + 2A^\alpha(y)\right)\right) |0\rangle \ , \tag{4.5}$$

where $\phi = \rho \exp i\theta\gamma^5$, $\langle 0| \rho |0\rangle = v$, and terms which do not give a zeroth order contribution have been dropped. In t'Hooft ξ-gauge there is no mixing between θ and A^μ, so in momentum space the leading term of (4.5) is

$$\begin{aligned} &\frac{1}{\pi}4v^2\epsilon_{\mu\nu}p^\mu g^2\left(\frac{g^{\nu\alpha} + \frac{(1-\xi)p^\nu p^\alpha}{\xi p^2 - 4g^2v^2}}{p^2 - 4g^2v^2}\right) \\ =&\frac{1}{\pi}\epsilon^{\mu\alpha}p_\mu\frac{4v^2g^2}{p^2-4g^2v^2} \ . \end{aligned} \tag{4.6}$$

At small p^2 this is just $-\frac{1}{\pi}\epsilon^{\mu\alpha}p_\mu$, giving the stated cancellation.

An almost identical cancellation occurs in the Schwinger model [11] with no fermion mass term. This model also has a massive spectrum. Furthermore the Ward identities are like those of the axial Higgs model, except that axial and vector labels are swapped:

$$\frac{\partial}{\partial y^\alpha} \langle 0| T^* \left(J_5^\mu(x)\ J^\alpha(y)\right) |0\rangle = 0 \ , \tag{4.7}$$

and

$$\frac{\partial}{\partial x^\mu} \langle 0| T^* \left(J_5^\mu(x)\ J^\alpha(y)\right) |0\rangle = -\frac{1}{\pi}\epsilon^{\mu\alpha}\frac{\partial}{\partial x^\mu}\delta(x-y) - \frac{1}{\pi}\langle 0| T^* \left(\epsilon_{\mu\nu}\partial^\mu A^\nu(x)\ J^\alpha(y)\right) |0\rangle \ . \tag{4.8}$$

In bosonized form [12] the last term of the latter ward identity can be written as

$$\langle 0|\, T^* \frac{e^2}{\pi\sqrt{\pi}}\phi(x)\ \frac{1}{\sqrt{\pi}}\epsilon^{\alpha\nu}\partial_\nu\phi\,|0\rangle\ \ , \tag{4.9}$$

where ϕ is a scalar field with mass $\frac{e}{\sqrt{\pi}}$. At momentum small compared to the coupling e, this becomes $\frac{1}{\pi}\epsilon^{\alpha\nu}p_\nu$ which cancels against the first (anomalous commutator) term of (4.8) . Thus the anomaly equation does not imply a massless pole.

5. Conclusion

The apparent paradox of an anomaly equation which is insensitive to particle masses has been resolved in 1+1 dimensions. The Higgs mechanism creates a gap, but also provides a means to cross the gap. In the presence of a localized background with Pontryagin number one, there is a bound fermion due to the winding Higgs background. This bound fermion acts as an "elevator" which carries charge across the gap. For uniform charge generating backgrounds, the Higgs degree of freedom prevents the existence of an adiabatic limit. In the dynamical case, the gauge boson becomes massive due to the Higgs. The gauge boson mass alters the anomalous ward identities in such a way that they do not imply the existence of a massless state. I believe the mechanisms described here should extend readily to $3+1$ dimensions and the standard model.

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6. Appendix A

Bogoliubov transformations for gauge theories: a paradox with $\epsilon^{\mu\nu}\gamma_\nu = \gamma^\mu\gamma^5$

In 1+1 dimensions $\gamma^\mu\gamma^5 = \epsilon_{\mu\nu}\gamma^\nu$ Therefore the 1+1 dimensional Dirac equation with an axial gauge field A^μ is equivalent to the Dirac equation with a background vector gauge

field V^μ where $V^\mu = \epsilon^{\mu\nu} A_\nu$. Thus it naively appears that if an axial gauge theory does not conserve vector charge, then neither does a vector gauge theory. Conversly if a vector theory does not conserve axial charge, it seems that an axial theory does not conserve axial charge either. Fortunately both these statements are not true.

The reason they are not true in a finite volume is that there is an ambiguity in doing Bogoliubov transformations. This ambiguity is removed by choosing either axial or vector gauge invariance. Consider the massless axial gauge theory in an $S_1 \otimes R_1$ space-time, and suppose charge is produced by a field strengh which vanishes at asymptotic times. The change in vector charge is equal to minus the change in the Chern-Simons number:

$$\Delta Q = \frac{g}{\pi} \Delta \oint dx^1 A_1 \ . \tag{6.1}$$

Therefore the gauge can be chosen so that A^μ vanishes in either the asymptotic past or the asymptotic future, but not both. I will call the Fermi field ψ^{in} or ψ^{out} depending on whether A^μ vanishes in the past or future. ψ^{in} can be expanded in terms of spinors which have definite momentum and frequency in the asymptotic past. Similiarly ψ^{out} can be expanded in terms of spinors which have definite momentum and frequency in the asymptotic future. Particle production is then determined from the Bogoliubov transformation relating the two sets of expansion coefficients.

Now suppose we were to consider the vector gauge theory with the backgrounds $V^\mu = \epsilon^{\mu\nu} A_\nu$. Suppose also that both the axial and vector field strenghths vanish at past and future times. If both field strengths vanish then $\epsilon^{\mu\nu}\partial_\mu A_\nu$ and $\partial_\mu A^\mu$ vanish and A^μ must be a constant. Consider a configuration with A^μ=0 in that past and $A^\mu = a^\mu$ in the future. The difference between an axial gauge theory and a vector gauge theory lies in the relation between ψ^{in} and ψ^{out}. For the axial theory

$$\psi^{out} = \exp(i g a_\mu x^\mu \gamma^5) \psi^{in} \ , \tag{6.2}$$

while for the vector theory

$$\psi^{out} = \exp(i g \epsilon_{\mu\nu} a^\mu x^\nu) \psi^{in} \ . \tag{6.3}$$

In light-cone coordinates, the two ψ^{out} fields are related by the transformation

$$\psi \rightarrow \exp(i g a_+ x^+ P_L + i g a_- x^- P_R) \psi \ . \tag{6.4}$$

This transformation changes the vector charge by an amount $g(a_+ - a_-)\frac{L}{2\pi}$ and the axial charge by an amount proportional to $g(a_+ + a_-)\frac{L}{2\pi}$, where L is circumference of S_1. Thus in a finite volume one finds the desired result that the axial theory produces only vector charge and the vector theory produces only axial charge.

The arguments above are not sufficient to show this result in an infinite volume. This is because in an infinite volume one can always find a gauge in which the vector potential vanishes in both the asymptotic past and asymptotic future *. For these gauges there is no difference between the out fields in the axial theory and the out fields in the vector theory: both are equal to the in field. However there is no equivalence between localized gauge invariant backgrounds in the axial theory and localized gauge invariant backgrounds in the vector theory provided that either vector charge or axial charge respectively are produced. If the axial and vector field strenghs are both localized, then $\epsilon^{\mu\nu}\partial_\mu A^\nu$ and $\partial_\mu A^\mu$ vanish outside some finite region of space-time. This means that A^μ must be a constant outside this region. The Pontryagin index for both the axial and the vector theory therefore vanishes. Note also that for the massive axial theory, a winding Higgs background has no Q.E.D. counterpart.

7. Appendix B

A No Pairing Theorem

The Euclidean equations of motion for the fermions of a spontaneously broken axial gauge theory possess the symetry $\psi \rightarrow \gamma^0\psi^*$. In this appendix I show that, in an adiabatic limit, this symetry does not yield independent solutions. To be precise, a solution of $\hat{K} f_0(x,\tau) = 0$ has the property that $\gamma^0 f_0^*(x,\tau) = \exp(i\alpha) f_0(x,\tau)$, where the phase α is a constant. The same is true for spinors annihilated by the adjoint operator $\hat{K}^\dagger$. Recall that the solution of $\hat{K} f_0(x,\tau) = 0$ in an adiabatic limit is

$$f_0(x,\tau) = \exp\left(\int_0^\tau d\tau' E_0(\tau')\right) \exp(i\beta(\tau))\chi_0(x,\tau) \ , \tag{7.1}$$

where $\chi_0(x,\tau)$ is an eigenfunction of the time dependent Hamiltonian for which the energy $E_0(\tau)$ crosses the gap. The Berry's phase $\beta(\tau)$ will turn out to be important to prevent pairing of zeromodes. At asymptotic positive x the magnitude of the Higgs field is v, and

* In a finite volume one is prevented from doing this by the gauge invariance of $\exp(ig\oint dx^1 A_1)$

one can always choose the gauge so that the phase of the Higgs field is independent of x. With this choice the bound state eigenfunctions of $H(\tau)$ at large x are of the form

$$\chi_0(x,\tau) = \begin{pmatrix} e^{ic(\tau)} \\ e^{-ic(\tau)} \frac{E_0(\tau)+i\kappa(\tau)}{\lambda v} \end{pmatrix} e^{-\kappa(\tau)x} e^{ia(\tau)} \ , \tag{7.2}$$

where

$$\kappa(\tau) = \sqrt{\lambda^2 v^2 - E_0^2(\tau)} \ , \tag{7.3}$$

$c(\tau)$ is the phase of Higgs, and $a(\tau)$ is an arbitrary phase. Therefore at large positive x

$$\gamma^0 \chi_0^*(x,\tau) = e^{-2ia(\tau)} \sqrt{\frac{E_0(\tau) - i\kappa(\tau)}{E_0(\tau) + i\kappa(\tau)}} \ \chi_0(x,\tau) \ . \tag{7.4}$$

It is easy show that the above relation holds at all x without knowing the exact form of the solution. If χ is an solution of

$$(\hat{H} - E)\chi = 0 \ , \tag{7.5}$$

then so is $\gamma^0\chi^*$, because

$$\gamma^0 \hat{H}^* \gamma^0 = \hat{H} \ . \tag{7.6}$$

Furthermore the eigenvalue equation (7.5) is linear and first order in x. Therefore if the relation (7.4) is true at any x, then it must be true at all x. We thus arrive at the result that

$$\gamma^0 f_0^*(x,\tau) = e^{-2i\beta(\tau)} e^{-2ia(\tau)} \sqrt{\frac{E(\tau) - i\kappa(\tau)}{E(\tau) + i\kappa(\tau)}} \ f_0(x,\tau) \tag{7.7}$$

It appears that there is a time dependent phase relation, but in fact the product of all the phases above is independent of τ. The Euclidean equations of motion are linear and first order in τ, and possess the symetry $\psi \to \gamma^0\psi^*$. Therefore if at some fixed τ

$$\gamma^0 f_0^*(x,\tau) = e^{i\alpha} f_0(x,\tau) \ , \tag{7.8}$$

then this relation must hold at all τ. The symetry which gives pairs of zeromodes in the massless theory fails to give pairs in the spontaneously broken theory.

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Figure Captions

Fig. 1. Illustration of the positive and negative chirality spectral flows of massless $3+1$ Q.E.D. which produce the axial anomaly. The solid lines indicate occupied states, while the dashed lines indicate empty states.

Fig. 2. Illustration of the spectral flows which produce the vector current anomaly in a spontaneously broken two dimensional axial gauge theory.

Fig. 3. A winding higgs field background for $\Delta Q = 1$. The time axis is vertical and the space axis is horizontal.